A useful application of Brun's irrationality criterion

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Transcendental number theory in a nutshell

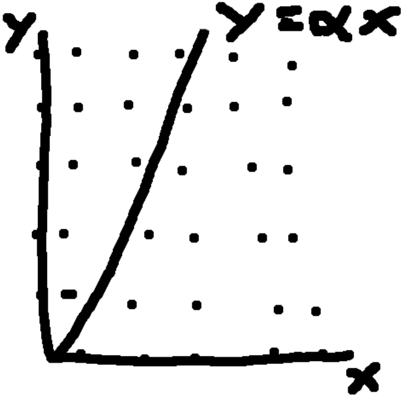
- Pick a real number α.
- With probability 1, α is irrational (i.e. $\neq a/b$).
- With probability 1, we won't be able to prove α is irrational.

Irrationality criteria

- Like testing for syphilis. But with irrationality. If α passes, then it's irrational.
- Don't work in general.
- $$\alpha = \lim_{n\to\infty} \frac{p_n}{q_n} \qquad \text{“quickly”}$$

Some geometry

- $\alpha = a/b$, $L : y = \alpha x$; $(nb, na) \in L$ for all $n \in \mathbb{Z}$.
- α is irrational, then the only integer point in L is $(0, 0)$.



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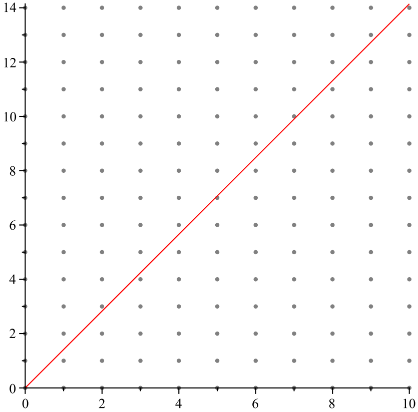

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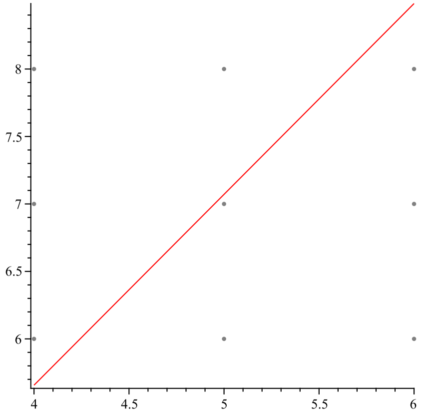


But wait, there's more!

- $\alpha = a/b$, $(m, n) \in \mathbb{Z}^2$

$$d((m,n), L) = \frac{|am - bn|}{\sqrt{a^2 + b^2}}.$$

- Now use...

Theorem (The fundamental theorem of transcendental number theory)

Let $k \in \mathbb{Z}$ satisfy $k \neq 0$, then

$$|k| \geq 1.$$

So...

$$d((m,n),L) = \frac{|am - bn|}{\sqrt{a^2 + b^2}}.$$

- Either $(m,n) \in L$,
- Or
$$d((m,n),L) \geq \frac{1}{\sqrt{a^2 + b^2}} = c(\alpha).$$

- Every $(m,n) \in \mathbb{Z}^2$ is either on L or at least $c(\alpha)$ away from it.

Our first irrationality criterion

- If $\exists\ (m, n) \in \mathbb{Z}^2$ arbitrarily close to – but not on – the line $y = \alpha x$, then α is irrational.

Not a million dollar question:

- How do we find these points?

Brun's irrationality criterion

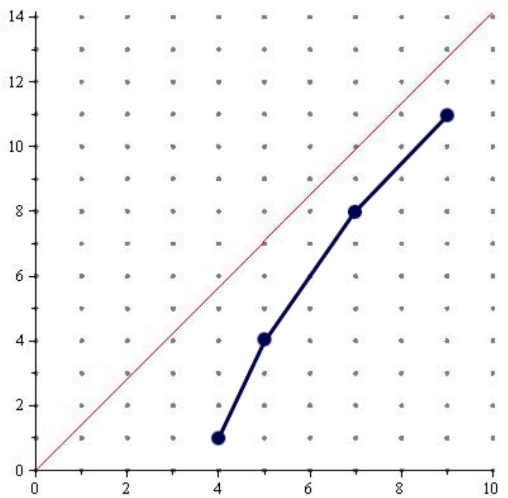
Theorem (Brun, 1910)

Let y_n and x_n be sequences of positive integers satisfying:

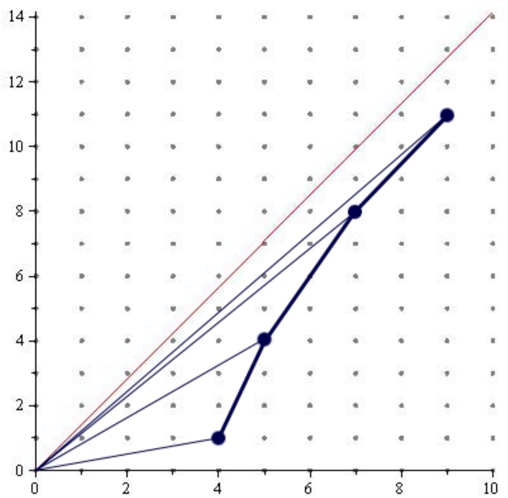
- $y_{n+1} > y_n$
- $x_{n+1} > x_n$
- $\frac{y_{n+1}}{x_{n+1}} > \frac{y_n}{x_n}$
- $\frac{y_{n+2}-y_{n+1}}{x_{n+2}-x_{n+1}} < \frac{y_{n+1}-y_n}{x_{n+1}-x_n}$
- $\lim_{n\to\infty} \frac{y_n}{x_n} = \alpha < \infty$

then α is irrational.

Proof by picture is a thing



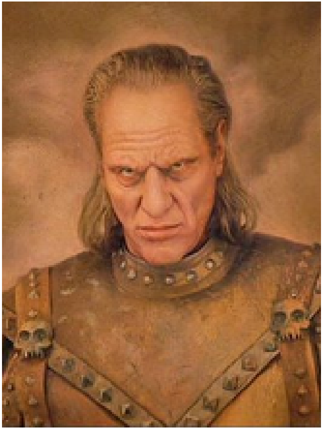
Proof by picture is a thing



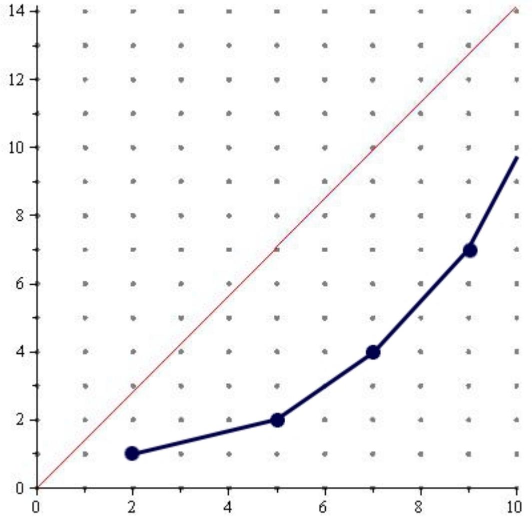
□

Viggo Brun (1972): "The theorem is simple but unfortunately not very useful"

Why's that, Viggo?



Viggo Brun (1972): "The picture will very often look like..."



0
2
4
6
8
10
12
14
0
2
4
6
8
10

And now for something completely different

- If $n \in \mathbb{N}^+$ is even then

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = q\pi^n, \quad q \in \mathbb{Q}^\times.$$

- In particular, $\zeta(2n)$ is irrational for all $n \in \mathbb{N}^+$.
- What about $\zeta(2n+1)$?

Conjecture

π, $\zeta(3)$, $\zeta(5)$, $\zeta(7)$,... are algebraically independent over $\mathbb{Q}$. In particular, $\zeta(2n+1)$ is irrational for all $n \geq 1$.

What do we know?

- Before 1978...nothing.

Journées Arithmétiques, 1978

- "Sur l'irrationalité de $\zeta(3)$"

Outlandish claims #1–3

- Let $a_0 = 0$, $a_1 = 6$, and for $n \geq 1$ let
$$a_{n+1} = \frac{(34n^3 + 51n^2 + 27n + 5)a_n - n^3 a_{n-1}}{(n+1)^3}.$$
- Let $b_0 = 1$, $b_1 = 5$, and for $n \geq 1$ let
$$b_{n+1} = \frac{(34n^3 + 51n^2 + 27n + 5)b_n - n^3 b_{n-1}}{(n+1)^3}.$$

Lemma (OC#1)

Let $\nu_n = 2(\operatorname{lcm}(1, 2, \ldots, n))^3$. Then $\nu_n a_n \in \mathbb{Z}$ for all $n \geq 0$.

Lemma (OC#2)

For all $n \geq 0$, $b_n \in \mathbb{Z}$.

Lemma (OC#3)

$a_n/b_n \to \zeta(3)$.

From now on,

$$\alpha_n = \nu_n a_n, \quad \beta_n = \nu_n b_n.$$

This looks familiar...

- $\alpha_{n+1} > \alpha_n$
- $\beta_{n+1} > \beta_n$
- $\frac{\alpha_{n+1}}{\beta_{n+1}} > \frac{\alpha_n}{\beta_n}$
- $\frac{\alpha_n}{\beta_n} \to \zeta(3)$.

Let

$$\delta_n = \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

If $\delta_n > \delta_{n+1}$ for all n then Brun's criterion applies.

Yes...

Let

$$\delta_n = \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

If $\delta_n > \delta_{n+1}$ for all n then Brun's criterion applies.

- $\delta_1 > \delta_2$

Yes...!

Let

$$\delta_n = \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

If $\delta_n > \delta_{n+1}$ for all n then Brun's criterion applies.

- $\delta_1 > \delta_2$
- $\delta_2 > \delta_3$

YES!

Let

$$\delta_n = \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

If $\delta_n > \delta_{n+1}$ for all n then Brun's criterion applies.

- $\delta_1 > \delta_2$
- $\delta_2 > \delta_3$
- $\delta_3 > \delta_4$

:(

Let

$$\delta_n = \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}.$$

If $\delta_n > \delta_{n+1}$ for all n then Brun's criterion applies.

- $\delta_1 > \delta_2$
- $\delta_2 > \delta_3$
- $\delta_3 > \delta_4$
- $\delta_4 < \delta_5$

The downward trends of delta n

Conjecture (Mingarelli, 2013)

The sequence δ_n contains arbitrarily long runs of strictly decreasing terms; in particular Brun's criterion can be applied!

Theorem (B., 2013)

Let $N \geq 2$ and let

$$n = (N! + N)! + N! + N,$$

then

$$\delta_n > \delta_{n+1} > \ldots > \delta_{n+N-1}.$$

Theorem (B., 2013)

This is not enough to apply Brun's criterion.

What's the problem?

- Contains arbitrarily long decreasing subsequences

$$\not\Rightarrow$$

contains strictly decreasing subsequence.

- Even if it did, that's not enough.
- If we pass to a subsequence of (α_n) and (β_n) then we change the definition of δ_n.
- Need an increasing sequence $n_k \to \infty$ such that

$$\delta_{n_k} := \frac{\alpha_{n_{k+1}} - \alpha_{n_k}}{\beta_{n_{k+1}} - \beta_{n_k}}$$

is strictly decreasing.

For $1 \leq m < n$ define

$$\delta(m,n) = \frac{\alpha_n - \alpha_m}{\beta_n - \beta_m}.$$

Can think of $\delta(m,n)$ as a function

$$\begin{array}{cccccc} 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & \dots \\ & 2,3 & 2,4 & 2,5 & 2,6 & \dots \\ & & 3,4 & 3,5 & 3,6 & \dots \\ & & & 4,5 & 4,6 & \dots \\ & & & & & \ddots \end{array} \to \mathbb{Q}$$

How is a subsequence like a nose?

Think of picking an increasing subsequence as picking an element (n_1, n_2) from row n_1 of the array, say $(1, 3)$:

$$\begin{array}{cccccc} 1,2 & \color{red}{1,3} & 1,4 & 1,5 & 1,6 & \dots \\ & 2,3 & 2,4 & 2,5 & 2,6 & \dots \\ & & 3,4 & 3,5 & 3,6 & \dots \\ & & & 4,5 & 4,6 & \dots \\ & & & & & \ddots \end{array}$$

Having picked (n_1, n_2) we go to row n_2 and pick (n_2, n_3):

$$\begin{array}{cccccc} 1,2 & \color{red}{1,3} & 1,4 & 1,5 & 1,6 & \dots \\ & 2,3 & 2,4 & 2,5 & 2,6 & \dots \\ & & \color{blue}{3,4} & \color{blue}{3,5} & \color{blue}{3,6} & \color{blue}{\dots} \\ & & & 4,5 & 4,6 & \dots \\ & & & & & \ddots \end{array}$$

For the goldfish in the audience

Remember, we want to pick our subsequence (n_k) so that

$$\delta(n_k, n_{k+1}) > \delta(n_{k+1}, n_{k+2})$$

for all k.

First apply δ to the whole array:

$$\begin{array}{cccccc} \delta(1,2) & \delta(1,3) & \delta(1,4) & \delta(1,5) & \delta(1,6) & \dots \\ & \delta(2,3) & \delta(2,4) & \delta(2,5) & \delta(2,6) & \dots \\ & & \delta(3,4) & \delta(3,5) & \delta(3,6) & \dots \\ & & & \delta(4,5) & \delta(4,6) & \dots \\ & & & & & \ddots \end{array}$$

Step 1 Show that for fixed m, as $n \to \infty$

$$\delta(m,n) \to \zeta(3).$$

Step 2 Show that for fixed m, for all sufficiently large n,

$$\delta(m,n) > \zeta(3).$$

Step 3 ???

Step 4 Profit.

First apply δ to the whole array:

$$\begin{array}{cccccc} \delta(1,2) & \delta(1,3) & \delta(1,4) & \delta(1,5) & \delta(1,6) & \dots \\ & \delta(2,3) & \delta(2,4) & \delta(2,5) & \delta(2,6) & \dots \\ & & \delta(3,4) & \delta(3,5) & \delta(3,6) & \dots \\ & & & \delta(4,5) & \delta(4,6) & \dots \\ & & & & & \ddots \end{array}$$

Step 1 Show that for fixed m, as $n \to \infty$

$$\delta(m,n) \to \zeta(3).$$

Step 2 Show that for fixed m, for all sufficiently large n,

$$\delta(m,n) > \zeta(3).$$

Step 3 Pick any (n_1, n_2) such that $\delta(n_1, n_2) > \zeta(3)$.

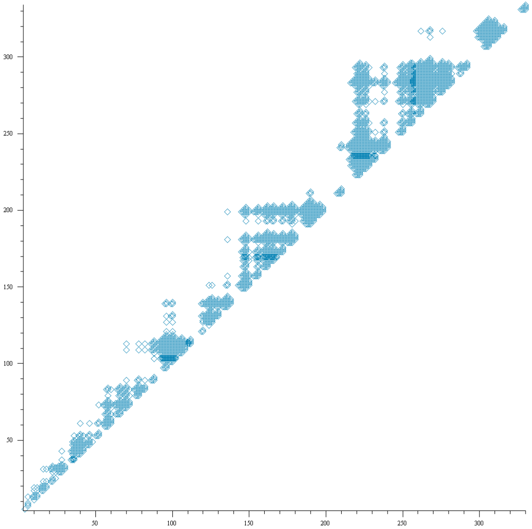
Step 4 Steps 1 and 2 guarantee existence of n_3 such that

$$\delta(n_1, n_2) > \delta(n_2, n_3) > \zeta(3).$$

Repeat!

- Step 1 is easy.
- Step 2 less so.
- The points where $\delta(m, n) < \zeta(3)$ seem to have some structure.

$\delta(m, n) < \zeta(3)$



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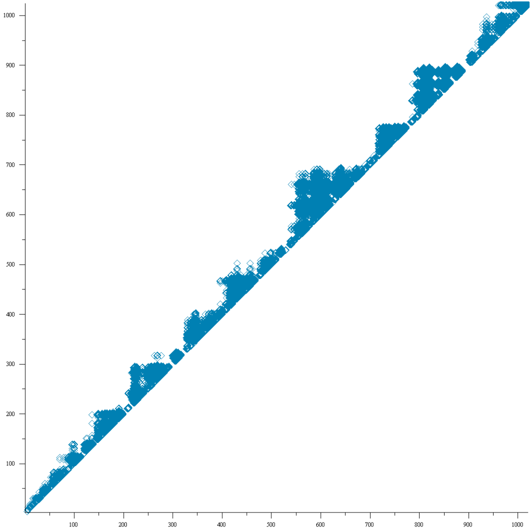

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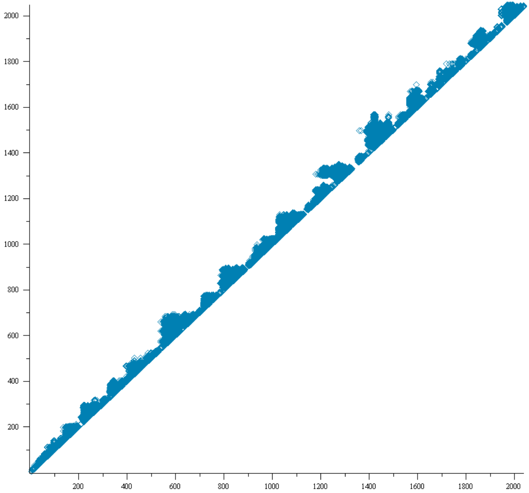
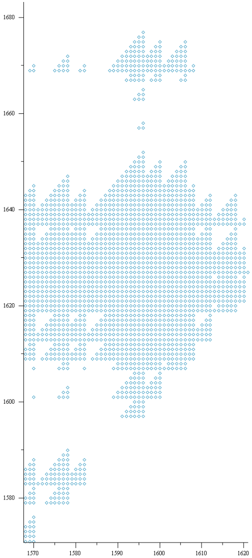


$\delta(m, n) < \zeta(3)$



Interesting, but irrelevant

- Structure seems to come from the prime numbers.
- Recall

$$\delta(m,n) = \frac{\nu_n a_n - \nu_m a_m}{\nu_n b_n - \nu_m b_m},$$

where

$$\nu_n = 2(\mathrm{lcm}(1,\ldots,n))^3.$$

- And

$$\mathrm{lcm}(1,\ldots,n) = \prod_{p\le n} p^{\lfloor \log n/\log p \rfloor}.$$

Also interesting, also irrelevant

Lemma

Let $1 \leq \ell < m < n$. *Then*

$$\frac{a_m - a_\ell}{b_m - b_\ell} > \frac{a_n - a_m}{b_n - b_m}.$$

Step 2

- We need to show for fixed m that $\delta(m, n) > \zeta(3)$ for all large n.
- Can reduce this to showing that $\nu_n = o(b_n)$, i.e.

$$\lim_{n \to \infty} \frac{\nu_n}{b_n} = 0.$$

Lemma (Hanson, 1972)

For large enough n, we have $\operatorname{lcm}(1, \ldots, n) \leq 3^n$.

Lemma

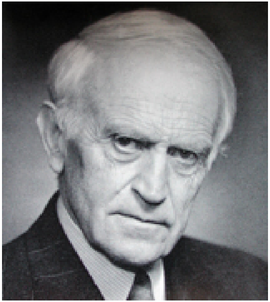
For large enough n, we have $b_n \geq 28^n$.

So eventually

$$\frac{\nu_n}{b_n} \leq 2\left(\frac{27}{28}\right)^n \to 0.$$

Corollary

There exists a sequence $n_k \to \infty$ *such that the sequences* $A_k = \nu_{n_k} a_{n_k}$ *and* $B_k = \nu_{n_k} b_{n_k}$ *satisfy the hypotheses of Brun's irrationality criterion.*



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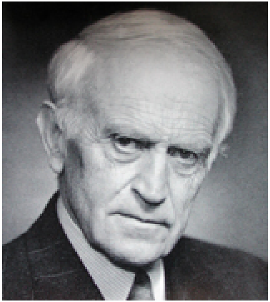

Here's one I prepared earlier

No, this doesn't apply to $\zeta(5)$ (or $\zeta(7)$, $\zeta(9)$, $\ldots$).